$$f(52) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$\Rightarrow \quad f(52) = 7071 + (1.4)(589) + \frac{(1.4)(0.4)}{2}(-57) + \frac{(1.4)(0.4)(-0.6)}{6}(-7)$$

$$\Rightarrow \quad f(52) = 7071 + 824.6 - 16 + 0.4 = 7880. \quad \text{Thus,} \quad \sin 52° = 7880.$$

Example 2. Given

x	1	2	3	4	5	6
$f(x)$	1	8	27	64	125	216

Estimate f (2.5).

Solution : We shall take, $x_0 = 1,\ h = 1,\ y_0 = 1$

Here $x = 2.5$ and also

$$x = x_0 + ph \quad \Rightarrow \quad p = \frac{x - x_0}{h} = \frac{2.5 - 1}{1} = 1.5$$

Now the forward difference table is

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

By Newton – Gregory forward interpolation formula, we have

$$f(2.5) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$\Rightarrow \quad f(2.5) = 1 + (1.5)(7) + \frac{(1.5)(0.5)}{2}(12) + \frac{(1.5)(0.5)(-0.5)}{6}(6)$$

$$\Rightarrow \quad f(2.5) = 1 + 10.5 + 4.5 - 0.375 = 15.625.$$

Example 3. Ordinates $f(x)$ of a normal curves in terms of standard deviation x are given as

x	1.00	1.02	1.04	1.06	1.08
$f(x)$	0.2420	0.2371	0.2323	0.2275	0.2227

Find the ordinate for standard deviation $x = 1.025$ by using Newton – Gregory forward interpolation formula.

Solution : Now the difference table is

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.00	2420				
		−49			
1.02	2371		1		
		−48		−1	
1.04	2323		0		1
		−48		0	
1.06	2275		0		
		−48			
1.08	2227				

Here, $x_0 = 1,\ h = 0.02$ and $y_0 = 2420$

Now, $x = 1.025$; $x = x_0 + ph \Rightarrow p = \dfrac{x - x_0}{h} = \dfrac{1.025 - 1}{0.02} = 1.25$

By Newton – Gregory forward interpolation formula, we have

$$10^4 f(1.025) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

$$= 2420 + (1.25)(-49) + \frac{(1.25)(0.25)}{2}(1)$$

$$+ \frac{(1.25)(0.25)(-0.75)}{6}(-1) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(1)$$

$$= 2420 - 61.25 + 0.15625 + 0.0390625 + 0.0170898$$

$$= 2358.9625 \quad \therefore \quad f(1.025) = 0.23589.$$

Example 4. From the table find the value of $e^{0.24}$

x	0.1	0.2	0.3	0.4	0.5
y	1.10517	1.22140	1.34986	1.49182	1.64872

Solution : Now the difference table is

x	$10^5 y$	Δ	Δ^2	Δ^3	Δ^4
0.1	110517				
		11623			
0.2	122140		1223		
		12846		127	
0.3	134986		1350		17
		14196		144	
0.4	149182		1494		
		15690			
0.5	164872				

Here $x_0 = 0.1,\ h = 0.1$ and $y_0 = 110517$

Now, $x = 0.24$ and

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{0.24 - 0.1}{0.1} = 1.4$$

By Newton – Gregory forward interpolation formula, we have

$$10^5 . f(0.24) = 110517 + (1.4)(11623) + \frac{(1.4)(0.4)}{2}(1223) + \frac{(1.4)(0.4)(-0.6)}{6}(127) + \frac{(1.4)(0.4)(-0.6)(-1.6)}{24}(17)$$

$$= 11057 + 16272.5 + 342.44 - 7.112 + 0.3808$$

$$= 127125.21. \quad \therefore \quad f(0.24) = 1.27125.$$

Example 5. Find the number of students from the following data who secured marks not more than 45.

Marks	30–40	40–50	50–60	60–70	70–80
No. of Students	35	48	70	40	22

Solution : First we shall prepare the cumulative frequency table.

Marks < (x)	40	50	60	70	80
No. of Sts. (y)	35	83	153	193	215

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	35				
		48			
50	83		22		
		70		–52	
60	153		–30		64
		40		12	
70	193		–18		
		22			
80	215				

Now, $x_0 = 40,\ x = 45,\ h = 10$

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

By Newton – Gregory forward interpolation formula, we have

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{6}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{24}\Delta^4 y_{40}$$

$$y_{45} = 35 + (0.5)(48) + \frac{(0.5)(-0.5)}{2}(22) + \frac{(0.5)(-0.5)(-1.5)}{6}(-52) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(64)$$

$$= 35 + 24 - 2.75 - 3.25 - 2.5 = 59 - 8.5 = 50.5 \approx 51 \text{ students.}$$

Example 6. Find the cubic polynomial which takes the following values.

x	0	1	2	3
$f(x)$	1	2	1	10

Now the difference table for x and $f(x)$ is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		1		
1	2		−2	
		−1		12
2	1		10	
		9		
3	10			

We shall take, $x_0 = 0$ and $p = \frac{x-0}{h} = x$ [since $h = 1$]

By using Newton – Gregory forward interpolation formula, we have

$$f(x) = f(0) + p\Delta f(0) + \frac{p(p-1)}{2!}\Delta^2 f(0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(0)$$

$$\Rightarrow \quad f(x) = 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$\Rightarrow \quad f(x) = 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$\Rightarrow \quad f(x) = 2x^3 - 7x^2 + 6x + 1.$$

Example 7. Estimate f(4.2) from the table :

x	0	2	4	6
$f(x)$	2	10	66	218

Solution : The difference table is

x	$f(x)$	Δ	Δ^2	Δ^3
0	2			
		8		
2	10		48	
		56		48
4	66		96	
		152		
6	218			

Now 4.2 is near the end of the table. Therefore we shall use Newton – Gregory backward interpolation formula.

Let us take, $x_n = 6,\ x = 4.2$ and $h = 2$

Now, $$p = \frac{x - x_n}{h} = \frac{4.2 - 6}{2} = -0.9$$

Newton – Gregory backward interpolation formula is

$$y_{4.2} = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

$$y_{4.2} = y_6 + (-0.9)\nabla y_n + \frac{(-0.9)(0.1)}{2}\nabla^2 y_6 + \frac{(-0.9)(0.1)(1.1)}{6}\nabla^3 y_6$$

$$y_{4.2} = 218 + (-0.9)(152) - (0.045)(96) - (0.0165)(48)$$

$$y_{4.2} = 218 - 136.8 - 4.32 - 0.792 = 76.088.$$

Example 8. The population of a town is as follows :

Year	1921	1931	1941	1951	1961	1971
Pop. in lakhs	20	24	29	36	46	51

Estimate the increase in population during the period 1955 to 1961.

Solution : The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1921	20					
		4				
1931	24		1			
		5		1		
1941	29		2		0	
		7		1		−9
1951	36		3		−9	
		10		−8		
1961	46		−5			
		5				
1971	51					

Here $\quad x_n = 1971,\ x = 1955$ and $h = 10$

$$\therefore \quad p = \frac{x - x_n}{h} \Rightarrow p = \frac{1955 - 1971}{10} = -1.6$$

Now, by Newton – Gregory backward interpolation formula

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots$$

$$y_{1955} = y_{1971} + (-1.6)\nabla y_{1971} + \frac{(-1.6)(-1.6+1)}{2}\nabla^2 y_{1971}$$

$$y_{1955} = 51 + (-1.6)(5) + \frac{(-1.6)(-0.6)}{2}(-5) + \frac{(-1.6)(-0.6)(0.4)}{6}(-8)$$

$$+ \frac{(-1.6)(-0.6)(0.4)(1.4)}{24}(-9) + \frac{(-1.6)(-0.6)(0.4)(1.4)(2.4)}{120}(-9)$$

$$y_{1955} = 51 - 8 - 2.4 - 0.512 - 0.202 + 0.097 = 39.983.$$

$\therefore$ increases in population during the period 1955 to 1961

$$= 46 - 39.983 = 6.017 \text{ lakhs.}$$

Example 9. From the following table find the value of tan 17°.

θ°	0	4	8	12	16	20	24
tanθ	0	0.0699	0.1405	0.2126	0.2867	0.3640	0.4452

Solution : The difference table is

x	$10^4 f(x)$	Δ	Δ^2	Δ^3	Δ^4
0	0				
		699			
4	699		7		
		706		8	
8	1405		15		−3
		721		5	
12	2126		20		7
		741		12	
16	2867		32		−5
		773		7	
20	3640		39		
		812			
24	4452				

Here, $\quad x_n = 24,\ x = 17$ and $h = 4$

$$p = \frac{x - x_n}{h} = \frac{17 - 24}{4} = -\frac{7}{4} = -1.75$$

Now by Newton – Gregory backward interpolation formula, we have

$$10^4 y_{17} = 4452 + (-1.75)\,812 + \frac{(-1.75)(-0.75)}{2}(39) + \frac{(-1.75)(-0.75)(0.25)}{6}(7) + \frac{(-1.75)(-0.75)(0.25)(1.25)}{24}(-5)$$

$$= 4452 - 1421 + 25.59 + 0.3828 - 0.0854 = 3056.887$$

Thus, $\tan 17° = 0.3056$.

EXERCISES

1. Given

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	2.68	3.04	3.38	3.68	3.96	4.21

Find $f(0.25)$.

2. Calculate $f(8.2)$ from the table.

x	8.0	8.5	9.0	9.5	10.0
$f(x)$	50	57	64	17	78

3. Given

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Estimate $f(3.5)$ and $f(7.5)$.

4. For the frequency distribution

Marks obtained	0–19	20–39	40–59	60–79	80–99
No. of candidates	41	62	65	50	17

Estimate the number of candidates who obtain less than 70 marks.

5. The following table gives the sales of a concern for the last five years. Estimate the sales for the year 1951.

years	1946	1948	1950	1952	1954
Sales (in thousands)	40	43	48	52	57

6. Find $f(142)$ from the following table.

x	140	150	160	170	180
$f(x)$	3.685	4.854	6.302	8.076	10.225

7. Estimate the value of tan (0.12) from the table given below.

x	0.10	0.15	0.20	0.25	0.30
$f(x)$	0.1003	0.1511	0.2027	0.2553	0.3093

8. Given

x	0.5	1.0	1.5	2.0
U_x	1	7.0	25	61

Find U_x at $x = 0.45$, at $x = 0.7$ and at $x = 2.1$.

9. Find $\log_{10}656$ from the table.

x	654	658	659	661
$\log_{10}x$	2.8156	2.8182	2.8189	2.8202

10. From the following table, estimate the number of students who obtain marks between 40 and 45.

Marks	30–40	40–50	50–60	60–70	70–80
No. of students	31	42	51	35	31

11. From the following table, estimate the number of persons earning wages between 60 and 70 rupees.

Wages (in Rs)	No. of persons (in thousands)
Below 40	250
40 – 60	120
60 – 80	100
80 – 100	70
100 – 120	50

12. Find the interpolating polynomial $f(x)$ which takes the values.

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

13. The first five terms of the sequences are 0, 0, 6, 12, 20. Find the sixth term and the pth term, $p > 7$.

14. The specific gravity of zinc sulphate solution of various concentrations at 15°C is given in the table. Obtain the specific gravity of 15.8% at 15°C.

conc.	10	12	14	16	18	20	22
sp.gr	1.059	1.073	1.085	1.097	1.110	1.124	1.137

15. The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth surface.

$x = ht$	100	150	200	250	300	350	400
$y = dis.$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when $x = 218$ ft and 410 ft.

16. From the following table, find $f(0.63)$.

x	0.30	0.40	0.50	0.60	0.70
$f(x)$	0.6179	0.6554	0.6915	0.7257	0.7580

17. Apply Newton's backward difference formula to the data below to obtain a polynomial of degree 4 in x.

x	1	2	3	4	5
$f(x)$	1	−1	1	−1	1

18. Apply Newton's backward difference formula to the data below to obtain a polynomial of degree 2 in x and hence find $f(2.5)$.

x	0	1	2	3
$f(x)$	1	3	7	13

19. The population of a country in decimal census were as follows.

year	1891	1901	1911	1921	1931
Pop.	46	66	81	93	101

20. The areas y of circles of different diameters x are given below.

x	80	85	90	95	100
y	5026	5674	6362	7088	7854

Calculate the area when $x = 98$.

21. From the following table of sines compute sin38°.

θ	0	10	20	30	40
sin θ	0	0.17365	0.34202	0.5	0.64279

22. The deflection y measured at various distances x from one end of a cantilever is given by

x	0.00	0.2	0.4	0.6	0.8	1.0
y	0.0000	0.0347	0.1173	0.2160	0.2987	0.3333

23. From the following table find y when $x = 1.85$ and 2.4.

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
y	5.474	6.050	6.686	7.389	8.166	9.025	9.974

24. Comput $e^{0.35}$ given

x	0	0.1	0.2	0.3	0.4
e^x	1.0000	1.1052	1.2214	1.3499	1.4918

25. Find $f(84)$ from the table.

x	40	50	60	70	80	90
$f(x)$	184	204	226	250	276	304

ANSWERS

1. 3.22 2. 53.8 3. 42.87 4. 199 5. 50.21 thousands
6. 3.899 7. 0.1205 8. 0.9; 2.34; 70.9 9. 2.8169
10. 17 11. 54 thousands 12. $x^2 + 2x + 3$ 13. 30, $p(p - 1)$
14. 1.0958 15. 15.7; 21.53 nautical miles 16. 0.7356
17. $y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$ 18. $x^2 + x + 1$; $f(2.5) = 9.75$
19. 97 thousands 20. 7542.5056 21. 0.61566 22. 0.3306
23. 6.36; 11.02 24. 1.4191 25. 287.

1.18 Interpolation with Unequal Intervals

In the previous section we have seen the interpolation formulae which are applicable only to equally spaced values of the argument. In this section we shall see two interpolation formulae for unequally spaced values of x.

1.Lagrange's formula for unequal intervals

Let $y = f(x)$ be a function whose values are $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced.

Since there are $(n + 1)$ pairs $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n.

Let this polynomial be

$$f(x) = A_0(x - x_1)(x - x_2) \ldots (x - x_n) + A_1(x - x_0)(x - x_2) \ldots (x - x_n) + A_2(x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots + A_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

where $A_0, A_1, A_2, \ldots, A_n$ are constants to be determined.

Putting $x = x_0$, $y = y_0$ we get

$$f(x_0) = A_0(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$\Rightarrow \quad A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)}$$

Putting $x = x_1, \; y = y_1$ we get

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)}$$

Similarly,

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_2) \ldots\ldots (x_n - x_{n-1})}$$

Substituting these values of $A_1, A_2, \ldots\ldots, A_n$, we get

$$f(x) = \frac{(x - x_1)(x - x_2) \ldots\ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots\ldots (x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2) \ldots\ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots\ldots (x_1 - x_n)} f(x_1)$$

$$+ \ldots\ldots + \frac{(x - x_0)(x - x_1) \ldots\ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots\ldots (x_n - x_{n-1})} f(x_n)$$

This formula is known as **Lagrange's interpolation formula for unequal intervals.**

Example 1. The following table gives the normal weights of babies during the first few months of life

Age in months	**2**	**5**	**8**	**10**	**12**
Weight in Kgs	**4.4**	**6.2**	**6.7**	**7.5**	**8.7**

Estimate by lagrange's method, the normal weight of a baby of 7 months old.

Solution : Let x = age of a baby (in months)

$f(x)$ = weight (in Kgs)

Then the data is

x	2	5	8	10	12
$f(x)$	4.4	6.2	6.7	7.5	8.7

By lagrange's formula, we have

$$f(x) = \frac{(x-5)(x-8)(x-10)(x-12)}{(2-5)(2-8)(2-10)(2-12)}(4.4)$$

$$+ \frac{(x-2)(x-8)(x-10)(x-12)}{(5-2)(5-8)(5-10)(5-12)}(6.2)$$

$$+ \frac{(x-2)(x-5)(x-10)(x-12)}{(8-2)(8-5)(8-10)(8-12)}(6.7)$$

$$+ \frac{(x-2)(x-5)(x-8)(x-12)}{(10-2)(10-5)(10-8)(10-12)}(7.5)$$

$$+ \frac{(x-2)(x-5)(x-8)(x-10)}{(12-2)(12-5)(12-8)(12-10)}(8.7)$$

$$f(7) = \frac{2(-1)(-3)(-5)}{(-3)(-6)(-8)(-10)}(4.4) + \frac{(5)(-1)(-3)(-5)}{(3)(-3)(-5)(-7)}(6.2)$$
$$+ \frac{(5)(2)(-3)(-5)}{(6)(3)(-2)(-4)}(6.7) + \frac{(5)(2)(-1)(-5)}{(8)(5)(2)(-2)}(7.5)$$
$$+ \frac{(5)(2)(-1)(-3)}{(10)(7)(4)(2)}(8.7)$$

$f(7) = -0.09 + 1.48 + 6.98 - 2.34 + 0.47 = 6.5$ Kg.

Example 2. Apply Lagrange's formula to find $f(5)$ and $f(6)$ given that $f(1) = 2, f(2) = 4$, $f(3) = 8, f(7) = 128$ and explain why the results differe from those obtained by $f(x) = 2^x$.

Solution : Here $x_0 = 1,\ x_1 = 2,\ x_2 = 2,\ x_3 = 7$

and $\quad f(x_0) = 2,\ f(x_1) = 4,\ f(x_2) = 8,\ f(x_3) = 128$

By lagrange's formula, we have

$$f(x) = \frac{(x-2)(x-3)(x-7)}{(1-2)(1-3)(1-7)} \cdot f(x_0) + \frac{(x-1)(x-3)(x-7)}{(2-1)(2-3)(2-7)} \cdot f(x_1)$$
$$+ \frac{(x-1)(x-2)(x-17)}{(3-1)(3-2)(3-7)} \cdot f(x_2) + \frac{(x-1)(x-2)(x-3)}{(7-1)(7-2)(7-3)} \cdot f(x_3)$$

$$f(x) = \frac{(x-2)(x-3)(x-7)}{-12}(2) + \frac{(x-1)(x-3)(x-7)}{5}(4)$$
$$+ \frac{(x-1)(x-2)(x-17)}{-8}(8) + \frac{(x-1)(x-2)(x-3)}{120}(128)$$

$$\therefore \quad f(5) = \frac{(3)(2)(-2)}{-12}(2) + \frac{(4)(2)(-2)}{5}(4) + \frac{(4)(3)(-2)}{-8}(8) + \frac{(4)(3)(2)}{120}(128)$$

$= 2 - 12.8 + 24 + 25.6 = 38.8$

$$\therefore \quad f(6) = \frac{(4)(3)(-1)}{-12}(2) + \frac{(5)(3)(-1)}{5}(4) + \frac{(5)(4)(-1)}{-8}(8) + \frac{(5)(4)(3)}{120}(128)$$

$= 2 - 12 + 20 + 64 = 74$

But actual values of $f(5)$ and $f(6)$ are $f(5) = 2^5 = 32$ and $f(6) = 2^6 = 64$.

The difference in values of $f(5)$ and $f(6)$ are due to the assumption of $f(x)$ as a polynomial, when it is an exponential function of the form 2^x.

Example 3. Using Lagrange's formula, prove that

$$y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}[(y_3 - y_1) - (y_{-1} - y_{-3})]$$

Solution : Let $x = -3, -1, 1, 3$.

$\therefore \quad y = f(x) = y_{-3},\ y_{-1},\ y_1,\ y_3$

Then from lagrange's formula, we have

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-3+1)(-3-1)(-3-3)} \cdot (y_{-3}) + \frac{(x-3)(x-1)(x-3)}{(-1+3)(-1-1)(-1-3)} \cdot (y_{-1})$$

$$+\frac{(x+3)(x+1)(x-3)}{(1+3)(1+1)(1-3)}\cdot(y_1)+\frac{(x+3)(x+1)(x-1)}{(3+3)(3+1)(3-1)}\cdot(y_3)$$

$$\therefore\quad y_0=-\frac{3}{48}(y_{-3})+\frac{9}{16}(y_{-1})+\frac{9}{16}(y_1)-\frac{3}{48}(y_{-3})$$

$$=-\frac{1}{16}(y_{-3})+\frac{8+1}{16}(y_{-1})+\frac{8+1}{16}(y_1)-\frac{1}{16}(y_{-3})$$

$$y_0=\frac{1}{2}(y_1+y_{-1})-\frac{1}{16}[(y_3-y_1)-(y_{-1}-y_{-3})].$$

Example 4. Use Legrange's interpolation formula to fit a polynomial to the data given below.

x	**0**	**1**	**3**	**4**
y	**−12**	**0**	**6**	**12**

Also find the value of y when $x = 2$.

Solution : For the given data, we have

$$x_0=0,\quad x_1=1,\quad x_2=3,\quad x_3=4$$

$$y_0=-12,\quad y_1=0,\quad y_2=6,\quad y_3=12$$

Thus the lagrange's formula is

$$y=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}\cdot(-12)+\frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}\cdot(0)$$

$$+\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}\cdot(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}\cdot(12)$$

$$=(x-1)(x-3)(x-4)-x(x-1)(x-4)+x(x-1)(x-3)$$

$$=(x-1)[x^2-7x+12-x^2+4x+x^2-3x]$$

$$=(x-1)(x^2-6x+12)=x^3-7x^2+18x-12$$

when $x=2,\quad y=y(2)=4$

EXERCISES

1. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

evaluate $f(9)$ using Lagrange's formula.

2. Certain corresponding values of x and $\log_{10}x$ are given below.

x	300	304	305	307
$\log_{10}x$	2.4771	2.4829	2.4843	2.4871

Find $\log_{10}310$ by Lagrange's formula.

3. Estimate the value of tan 32° from the data.

x	30°	35°	45°	50°	60°
tan x	0.5773	0.7002	1	1.1918	1.7320

Using Lagrange's formula

4. Use Lagrange's formula to find the value of $f(x)$ at $x = 6$ from the data.

x	3	7	9	10
$f(x)$	168	120	72	63

5. Use Lagrange's formula to find the form of $f(x)$, given

x	0	2	3	6
$f(x)$	648	704	729	792

6. Given the following table of values

x	0.3	0.5	0.6	0.8
$f(x)$	−0.91	−0.75	−0.64	−0.36

Estimate the value of $f(0.7)$ using Lagrange's formula.

ANSWERS

1. 810 2. 2.4786 3. 0.6249 4. 147 5. $648 + 30x - x^2$ 6. −0.51.

1.19 Inverse Interpolation

So far we have been finding the values of y corresponding to a certain value of x, when a set of values of x and y are given.

The process of estimating the value of x for a given value of y is called the **inverse interpolation.**

We have Lagrange's interpolation formula,

$$y = \frac{(x - x_1)(x - x_2)......(x - x_n)}{(x_0 - x_1)(x_0 - x_2)......(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)......(x - x_n)}{(x_1 - x_0)(x_1 - x_2)......(x_1 - x_n)} y_1 + ... + \frac{(x - x_0)(x - x_1)......(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)......(x_n - x_{n-1})} y_n$$

This formula is merely a relation between two variables x and y, either of which may be taken as the independent variable. Thus by interchanging x and y in the above formula we obtain,

$$x = \frac{(y - y_1)(y - y_2)......(y - y_n)}{(y_0 - y_1)(y_0 - y_2)......(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)......(y - y_n)}{(y_1 - y_0)(y_1 - y_2)......(y_1 - y_n)} x_1$$

$$+ \ldots\ldots + \frac{(y - y_0)(y - y_1)\ldots\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots\ldots(y_n - y_{n-1})} x_n$$

which gives the value of x corresponding to a given value of y. This formula is known as **Lagrange's inverse interpolation formula.**

Example 1. Given

x	1	3	4
y	4	12	19

find x corresponding to $y = 7$.

Solution : Here, $x_0 = 1$, $y_0 = f(1) = 4$

$x_1 = 3$, $y_1 = f(3) = 12$

$x_2 = 4$, $y_2 = f(4) = 19$

We have to find x such that $y = f(x) = 7$

By Lagrange's inverse interpolation formula, we have

$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)} \cdot 1 + \frac{(7-4)(7-19)}{(12-4)(12-19)} \cdot 3 + \frac{(7-4)(7-12)}{(19-4)(19-12)} \cdot 4 \Rightarrow$$

$$x = \frac{60}{120} + \frac{108}{56} - \frac{60}{105}$$

$$\Rightarrow \quad x = 0.5 + 1.92875 - 0.57143 = 1.85714$$

Example 2. Find a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, $f(42) = 18$, by Lagrange's inverse interpolation formula.

Solution : Here, $x_0 = 30$, $y_0 = f(30) = -30$

$x_1 = 34$, $y_1 = f(34) = -13$

$x_2 = 38$, $y_2 = f(38) = 3$

$x_3 = 42$, $y_3 = f(42) = 18$

We have to find x such that $y = f(x) = 0$

By Lagrange's inverse interpolation formula, we have

$$x = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)} \cdot 30 + \frac{(0+30)(0-3)(0-18)}{(-30+30)(-30-3)(-13-18)} \cdot 34 + \frac{(0+30)(0+3)(0-18)}{(3+30)(3+13)(3-18)} \cdot 38 + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)} \cdot 42$$

$$x = \frac{(13)(-3)(-18)(30)}{(-17)(-33)(-48)} + \frac{(30)(-3)(-18)(34)}{(17)(-16)(-31)}$$

$$+ \frac{(30)(13)(-18)(38)}{(33)(16)(-15)} + \frac{(30)(13)(-3)(42)}{(48)(31)(15)}$$

$\Rightarrow$ $$x = -0.7820 + 6.5322 + 33.6818 - 2.2016 = 37.2304$$

This is an approximate root of the equation $f(x) = 0$.

EXERCISES

1. Find the value of x when $f(x) = 15$ by applying lagrange's method, given

x	5	6	9	11
$f(x)$	12	13	14	16

2. Given

x	2	5	9	11
y	10	12	15	19

Find x when $y = 16$.

3. Find the value of x when $y = 16$.

x	2	4	6	8	10
y	7	15	11	14	12

4. Find x corresponding to $y = 0.163$ from the data.

x	80	82	84	86	88
y	0.134	0.154	0.176	0.200	0.227

5. Solve the equation $f(x) = 0$, given

x	0	0.1	0.2	0.3	0.4
$f(x)$	1	0.80484	0.61873	0.44082	0.27032

ANSWERS

1. 11.5 2. 9.97143 3. 3.45 4. 82.72 5. 0.5671.

1.20 Numerical Integration

Numerical integration is a process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$. If the integrand is a function of a single-valued, then the numerical integration is known as **Quadrature.**

First we shall obtain the general quadrature formula for equidistant ordinates and then we shall deduce from this the other quadrature rules.

1. Newton – Cote's Quadrature formula

Let $$I = \int_a^b f(x)\, dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots, y_n$ for $x = x_0, x_1, x_2, \ldots, x_n$. Divide the interval (a, b) into n sub-intervals of width h, so that

$$x_0 = a, \; x_1 = x_0 + h, \; x_2 = x_0 + 2h, \; \ldots, \; x_n = x_0 + nh = b$$

Now, $$I = \int_{x_0}^{x_0 + nh} f(x)\, dx$$

Let $x = x_0 + rh \Rightarrow dx = h\,dr$. Also when $x = x_0,\ r = 0$ and when $x = x_0 + nh,\ r = n$

$$\therefore \quad I = \int_0^n f(x_0 + rh)\, h\, dr$$

Now by Newton's forward interpolation formula, we have

$$I = h\int_0^n \left[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + ...\right] dr$$

Integrating term by term, we get

$$I = h\left[y_0 r + \frac{r^2}{2}\Delta y_0 + \frac{1}{2!}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{3!}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + ...\right]_0^n$$

$$I = nh\left[y_0 + \frac{n}{2}(\Delta y_0) + \frac{1}{2!}\left(\frac{n^2}{3} - \frac{n}{2}\right)(\Delta^2 y_0) + \frac{1}{3!}\left(\frac{n^3}{4} - n^2 + n\right)(\Delta^3 y_0) + ...\right]$$

This formula is known as **Newton-Cote's quadrature formula** or a **general Quadrature formula.**

Now we shall consider other Quadrature rules by taking $n = 1, 2, 3, ...$

1. Trapezoidal Rule

By taking $n = 1$ in the general Quadrature formula and neglecting terms containing $\Delta^2 y_0, \Delta^3 y_0, ...$ we get

$$\int_{x_0}^{x_0 + h} f(x)\, dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_2)\right] = \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly,

$$\int_{x_0 + h}^{x_0 + 2h} f(x)\, dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right] = \frac{h}{2}\left[y_1 + y_2\right]$$

....

....

$$\int_{x_0 + (n-1)h}^{x_0 + nh} f(x)\, dx = = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding these n integrals, we obtain,

$$\int_{x_0}^{x_0 + nh} f(x)\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

$= \frac{h}{2}$ [(Sum of first and last ordinates) + 2 (sum of the remaining)]

This formula is known as **Trapezoidal Rule** for numerical integration.

2. Simpson's one–third rule

By taking $n = 2$ in the general Quadrature formula and neglecting terms containing $\Delta^3 y_0, \Delta^4 y_0, ...$ we get

$$\int_{x_0}^{x_0+2h} f(x)\, dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0 \right]$$

But, $\Delta y_0 = y_1 - y_0,\ \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$

$$\therefore \quad \int_{x_0}^{x_0+2h} f(x)\, dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2 \right]$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\, dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4 \right]$$

....

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n \right]$$

[Here n is even]

Adding these n integrals, we obtain, when n is even,

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{3}\Big[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1})$$

$$+ 2(y_2 + y_4 + ... + y_{n-2}) \Big]$$

$= \frac{h}{3}$ [(Sum of first and last ordinates) + 4(sum of even ordinates)

+ 2 (sum of the remaining odd ordinates)]

This formula is known as **Simpson's one–third** rule or simply **Simpson's rule.**

To apply this rule the given interval must be divided into even number of equal sub-intervals.

3. Simpson's one–eigth rule

By taking $n = 3$ in the general Quadrature formula and neglecting terms containing $\Delta^4 y_0, \Delta^5 y_0, ...$ we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}(\Delta y_0) + \frac{3}{4}(\Delta^2 y_0) + \frac{1}{8}(\Delta^3 y_0)\right]$$

[Since $\Delta y_0 = y_1 - y_0,\ \Delta^2 y_0 = y_2 - 2y_1 + y_0,\ \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$]

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_1 + 3y_4 + 3y_5 + y_6\right]$$

....

....

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

[Here n is a multiple of 3]

Adding these n integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3}{8}h\Big[(y_0 + y_n)$$
$$+ 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-4} + y_{n-2} + y_{n-1})$$
$$+ 2(y_3 + y_6 + \ldots\ldots + y_{n-3})\Big]$$

This formula is known as **Simpson's three–eighth rule.**

This formula is applicable only when n is a multiple of 3.

Example 1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

(i) **Trapizoidal rule** (ii) **Simpson's $\frac{1}{3}$ rule** (iii) **Simpson's $\frac{3}{8}$ rule**

Solution : Divide the interval (0, 6) into six parts each of width $h = 1$. Let $f(x) = \frac{1}{1+x^2}$.

Then

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027

(i) **By Trapizoidal rule**

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)]$$

$$= \frac{1}{2}[1.027 + 1.7946] = 1.4108$$

(ii) Simpson's $\frac{1}{3}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= \frac{1}{3}[1.027 + 2.554 + 0.5176] = 1.3662$$

(iii) Simpson's $\frac{3}{8}$ rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)]$$

$$= \frac{3}{8}[1.027 + 2.3919 + 0.2] = 1.3571$$

Example 2. Calculate $\int_4^{5.2} \log x \, dx$ by using

(i) Trapizoidal rule **(ii) Simpson's $\frac{1}{3}$ rule** **(iii) Simpson's $\frac{3}{8}$ rule**

given

x	4.0	4.2	4.4	4.6	4.8	5	5.2
$\log x$	1.38629	1.43508	1.48160	1.52605	1.56861	1.60943	1.64865

(i) By Trapizoidal rule. Here $n = 6$ and $h = 0.2$

$$\int_4^{5.2} \log x \, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}[(1.38629 + 1.64865) + 2(1.43508 + 1.48160 + 1.52605 + 1.56861 + 1.60943)]$$

$$= \frac{0.2}{2}[3.034953 + 15.24154] = (0.1)(18.27649) = 1.82765.$$

(ii) Simpson's $\frac{1}{3}$ rule

$$\int_4^{5.2} \log x \, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.2}{3}[(1.38629 + 1.64865) + 4(1.43508 + 1.52605 + 1.60943) + 2(1.48160 + 1.56861)]$$

$$= \frac{0.2}{3}[3.034953 + 18.28224 + 6.10042]$$

$$= \frac{0.2}{3}(27.41761) = 1.82784$$

(iii) Simpson's $\frac{3}{8}$ rule

$$\int_4^{5.2} \log x \, dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3(0.2)}{2}[(1.38629 + 1.64865) + 3(1.43508 + 1.48160$$

$$+ 1.56861 + 1.60943) + 2(1.52605)]$$

$$= \frac{0.6}{8}[3.034953 + 18.28416 + 3.0521]$$

$$= \frac{0.6}{8}[24.371213] = 1.82784.$$

$$= \frac{0.6}{10}[30.464] = 1.82784.$$

Example 3. Evaluate $\int_0^1 e^x dx$ approximately in steps of 0.2 using by using Trapizoidal rule.

Solution : Let $y_x = e^x$ and $h = 0.2$.

The values of x and y_x are given below.

x	0	0.2	0.4	0.6	0.8	1.0
y_x	1 y_0	1.2114 y_1	1.4918 y_2	1.8221 y_3	2.2255 y_4	2.7183 y_5

By Trapizoidal rule

$$\int_0^1 e^x \, dx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}[(1 + 2.7183) + 2(1.2114 + 1.4918 + 1.8221 + 2.2255)]$$

$$= (0.1)[3.7183 + 13.5014]$$

$$= (0.1)[17.2399] = 1.72197.$$

Example 4. Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^3 \frac{dx}{1+x}$ by dividing the interval (0, 3) into six equal parts.

Solution : Let $y = \frac{1}{1+x}$ and $h = 0.5$.

The values of x and y are tabulated below.

x	0	0.5	1.0	1.5	2	2.5	3.0
y	1	0.667	0.500	0.400	0.333	0.286	0.250

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^3 \frac{dx}{1+x} = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.5}{3}[(1 + 0.250) + 4(0.667 + 0.400 + 0.286)$$

$$+ 2(0.500 + 0.333)]$$

$$= \frac{0.5}{3}[1.250 + 5.412 + 1.666] = \frac{0.5}{3}[8.328] = 1.388$$

Example 5. Using Simpson's $\frac{3}{8}$ rule obtain an approximate value of $\int_0^{0.3} (2x - x^2)^{1/2}\, dx$.

Solution : Let $y = (2x - x^2)^{1/2}$ and $h = 0.05$. Thus $n = 6$.

Now,

x	0	0.05	0.10	0.15	0.20	0.25	0.30
y	0	0.3122	0.4359	0.5268	0.6000	0.6614	0.7141

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_0^{0.3} (2x - x^2)^{1/2} dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3(0.05)}{8}[0 + 0.7141 + 3(0.3122 + 0.4359 + 0.6000 + 0.6614)$$

$$+ 2(0.5268)]$$

$$= \frac{0.15}{8}[0.7141 + 6.0825 + 1.0536] = \frac{0.15}{8}[7.7962] = 0.14617.$$

Example 6. A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x = 0$ and $x = 1$ and a curve through the points with the following coordinates.

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Solution : The volume of the solid generated is given by

$$\int_0^1 \pi y^2 \, dx$$

By Simpson's $\frac{1}{3}$ rule, we have (here $h = 0.25$)

$$\int_0^1 \pi y^2 \, dx = \pi \frac{h}{3}\left[(y_0^2 + y_4^2) + 4(y_1 + y_3^2) + 2(y_2^2)\right]$$

$$= \frac{0.25}{3}\pi\{1 + (0.8415)^2 + 4[(0.9896)^2 + 0.9089)^2] + 2(0.9589)^2\}$$

$$= \frac{(0.25)(3.1416)}{3}[1.7081 + 4(0.9793 + 0.8261) + 1.8389]$$

$$= (0.2618)[1.7081 + 7.2216 + 1.8389]$$

$$= (0.2618)(10.7686) = 2.8192.$$

EXERCISES

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using

   (i) Trapizoidal rule (ii) Simpson's $\frac{1}{3}$ rule (iii) Simpson's $\frac{3}{8}$ rule

2. Evaluate $\int_0^6 \frac{dx}{1+x}$ by using

   (i) Trapizoidal rule (ii) Simpson's $\frac{1}{3}$ rule (iii) Simpson's $\frac{3}{8}$ rule

3. Evaluate $\int_1^7 f(x)\, dx$ by Trapizoidal rule using the following table.

x	1	2	3	4	5	6	7
y	2.105	2.808	3.614	4.604	5.857	7.451	9.467

4. Use Trapizoidal rule to evaluate $\int_4^{5.2} y_x\, dx$, given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y_x	1.386	1.435	1.482	1.526	1.569	1.609	1.649

5. Using Trapizoidal rule evaluate $\int_0^6 y_x\, dx$, given

x	0	1	2	3	4	5	6
y	0.146	0.161	0.176	0.190	0.204	0.217	0.230

6. Evaluate $\int_0^1 \frac{dx}{1+x}$ by Trapizoidal rule by considering eight subintervals of the interval [0, 1]. Hence find an approximate value of log2.

7. Evaluate $\int_{-3}^3 x^4\, dx$ with $h = 1$, by Trapizoidal rule.

8. Evaluate $\int_1^2 \frac{dx}{x}$ with $h = 5$, by Trapizoidal rule.

9. Using Simpson's $\frac{1}{3}$ rule evaluate $\int_1 \frac{dx}{1+x}$ by dividing (0, 1) into 8 equal parts.

10. By using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval [0, 1] into six equal parts. Hence find an approximate value of π

11. By using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^5 \frac{dx}{4x+5}$ by dividing the range into 10 equal parts. Hence find an approximate value of log5.

12. By Simpson's $\frac{1}{3}$ rule evaluate $\int_0^2 f(x)\, dx$, given

x	0.0	0.5	1.0	1.5	2.0
$f(x)$	0.399	0.352	0.242	0.129	0.054

13. Use Simpson's $\frac{1}{3}$ rule to evaluate $\int_0^6 \frac{dx}{(1+x)^2}$ correct to 3 places of decimals in steps of 1 unit.

14. A curve $y = f(x)$ passes through the points (x, y) given in the following table,

x	0	0.25	0.5	0.75	1
y	1	4	8	4	1

By using Simpson's $\frac{1}{3}$ rule, find the volume generated when the curve is revolved about the x - axis.

15. The following tables gives 6 values of an independent variable x and the corresponding values of $y = f(x)$

x	0	1	2	3	4	5	6
y	0.46	0.161	0.176	0.190	0.204	0.217	0.230

16. Evaluate $\int_0^1 \frac{x}{1+x^2}\, dx$ by using Simpson's $\frac{3}{8}$ rule, dividing the interval into 3 equal parts. Hence find an approximate value of $\log\sqrt{2}$.

17. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$ rule, dividing the interval into 6 equal parts.

18. Evaluate $\int_4^{5.2} \log x\, dx$ by using Simpson's $\frac{3}{8}$ rule, given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

19. By using $\frac{3}{8}$ rule with $h = 0.2$, find the approximate area under the curve $y = \frac{x^2-1}{x^2+1}$ between the ordinate $x = 1$ and $x = 2.8$.

20. Estimate the length of the arc of the curve $3y = x^3$ from (0, 0) to (1, 3) using Simpson's $\frac{1}{3}$ rule taking 8 subintervals. [Hint : $s = \int_0^1 \sqrt{1 + (dy/dx)^2}\, dx$]

ANSWERS

1.	(i) 1.4108		(ii) 1.3662		(iii) 1.3571		(iv) 1.4056				
2.	1.9588 (appx)	3.	30.12	4.	1.8277	5.	1.136	6.	0.694125		
7.	115	8.	0.6921	9.	0.6932	10.	0.785397, 3.141588				
11.	1.61	12.	0.477	13.	0.874	14.	6.8095	15.	1.13625		
16.	0.348077	17.	1.357082	18.	1.8278	19.	0.9152	20.	1.0893.		

1.21 Numerical Solution of Ordinary Differential Equations

In this section we deal with some simple numerical methods of solving first order ordinary differential equations.

We consider few methods of solving the differential equations of the form

$$\frac{dy}{dx} = f(x, y)$$

subjected to the condition of the form $y(x_0) = y_0$

Here the initial condition $y(x_0) = y_0$ is specified at the point x_0. Such problems in which the initial conditions are given at the initial point only are called **initial value problems.**

1.22 Taylor's Series method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0 \qquad \text{..... (1)}$$

Differentiating (1) w.r.t x we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

i.e. $$y'' = f_x + f_y f'$$

Differentiating successively, we obtain y''', y^{iv}, etc.

Now by putting $x = x_0$ and $y = y_0$, we get the values of $(y')_0$, $(y'')_0$, $(y''')_0$ etc.

Now we have the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \text{.....}$$

This gives us the Taylor series solution at the point x in a neighbourhood of the point x_0.

Example 1. Find by Taylors series method the value of y at $x = 0.2$ correct to four decimal places if $y(x)$ satisfies

$$\frac{dy}{dx} = x - y^2 \quad \text{and} \quad y(0) = 1$$

Solution : We have

$y' = x - y^2$ $\qquad (y')_0 = -1$

$y'' = 1 - 2yy'$ $\qquad (y'')_0 = 1 - 2(-1) = 3$

$y''' = -2yy'' - 2(y')^2$ $\qquad (y''')_0 = -6 - 2 = -8$

$y^{iv} = -2y''' - 2y'y'' - 4y'y''$ $\qquad (y^{iv})_0 = (-2)(-8) - 6(-1).3 = 34$

$y^{v} = -2[yy^{iv} + 4y'y''' + 3(y'')^2]$ $\qquad (y^{v})_0 = -2[34 - 4(-1)(-8) - 6(9) = -186$

The Taylor's series solution is given by

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + ...$$

$$\Rightarrow \qquad y = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + ...$$

Putting $x = 0.2$, we get

$$y = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 - \frac{31}{20}(0.2)^5 + ...$$

$$\Rightarrow \qquad y = 1 - 0.2 + 0.06 - 0.010666 + 0.002266 - 0.000496$$

$$\Rightarrow \qquad y = 0.8511 \text{ (correct to four decimal places).}$$

In the Taylor series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + ...$$

Putting $x = x_1 = x_0 + h$, we get

$$y(x_0 + h) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + ...$$

This is the Taylor series solution of the given equation at the point $x_1 = x_0 + h$. Here $y_0, y_0', y_0'', ...$ are the values of $y, y', y'' .....$ at $x = x_0$.

Example 2. Using Taylors series find the solution of $x\frac{dy}{dx} = x - y$, $y(2) = 2$ at $x = 2.1$ correct to five decimal places.

Solution : Here $x_0 = 2$ and $y_0 = y(2) = 2$

The given differential equation is

$$y' = \frac{x - y}{x} = 1 - \frac{y}{x}$$

$y'(x) = 1 - \frac{y}{x}$ $\qquad y'(2) = 1 - \frac{1}{1} = 0$

$$y''(x) = -\frac{y'}{x} + \frac{y}{x^2} \qquad y''(2) = 0 + \frac{2}{4} = \frac{1}{2}$$

$$y'''(x) = \frac{y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} \qquad y'''(2) = -\frac{1}{4} + 0 - \frac{4}{8} = -\frac{3}{4}$$

$$y^{iv}(x) = -\frac{y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4} \qquad y^{iv}(2) = \frac{3}{8} + \frac{3}{8} - 0 + \frac{12}{16} = \frac{3}{2}$$

Now, the Taylor's series solution we have

$$y(2+h) = y(2) + \frac{h}{1!}y'(2) + \frac{h^2}{2!}y''(2) + \frac{h^3}{3!}y'''(2) + \ldots..$$

$$\Rightarrow \quad y(2+h) = 2 + h.0 + \frac{h^2}{2}.\frac{1}{2} + \frac{h^3}{6}\left(-\frac{3}{4}\right) + \frac{h^4}{24}\left(\frac{3}{2}\right) + \ldots..$$

$$\Rightarrow \quad y(2+h) = 2 + \frac{h^2}{4} - \frac{h^3}{8} + \frac{h^4}{16} + \ldots..$$

Putting $h = 0.1$, we get

$$y(2.1) = 2 + \frac{(0.1)^2}{4} - \frac{(0.1)^3}{8} + \frac{(0.1)^4}{16} + \ldots..$$

$$\Rightarrow \quad y(2.1) = 2 + 0.0025 + 0.000125 + 0.0000062 = 2.0026312.$$

Example 3. Using Taylor's series method, find $y(0.1)$ correct to 3 decimal places $\frac{dy}{dx} + 2xy = 1$, given $y_0 = 0$.

Solution : We have $\frac{dy}{dx} + 2xy = 1 \quad \Rightarrow \quad y' = 1 - 2xy$

Now, $\quad y' = 1 - 2xy \qquad y'(0) = 1$

$y'' = -2(x + xy') \qquad y''(0) = 0$

$y''' = -2(xy'' + 2y') \qquad y'''(0) = -4$

By data, $x_0 = 0, \quad y_0 = 0$ and $h = 0.1$

Now, the Taylor's series solution is,

$$y(x_0 + h) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots..$$

i.e $\quad y(0.1) = 0 + 0.1 + \frac{(0.1)^2}{2!} \times 0 + \frac{(0.1)^3}{2!} \times (-4) = 0.09933$

EXERCISES

1. Using Taylor's series method, compute the solution of $\frac{dy}{dx} = x + y$, $y(0) = 1$ at $x = 0.2$ correct to three decimal places.

2. Solve $\frac{dy}{dx} = y^2 + x$, $y(0) = 1$, using Taylor's series method and compute $y(0.1)$ and $y(0.2)$.

3. Solve $\frac{dy}{dx} = x^2 + y^2, y(0) = 0$, by Taylor's series method and compute $y(1)$.

4. Solve $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$, by Taylor's series method and compute $y(0.1)$ and $y(0.2)$.

5. Solve $\frac{dy}{dx} = 2x + 3y, y(0) = 1$, by Taylor's series method for $x = 0.1\ (0.1)0.3$.

ANSWERS

1. 1.243 2. 1.1164, 1.2725 3. 0.3502 4. 0.90033, 0.80277
5. 1.355, 1.8559, 2.55161.

1.23 Euler's Method

Consider the equation

$$\frac{dy}{dx} = f(x, y) \quad \text{..... (1)}$$

with initial condition $y = y_0$ at $x = x_0$

Let $y = y(x)$ be the actual solution of the equation. Let h be the interval of differencing.
We have by mean value theorem,

$$y(x + h) = y(x) + h y'(x) \quad \text{..... (2)}$$

Taking $x = x_0$, we get

$$y(x_0 + h) = y(x_0) + h y'(x_0)$$

$$\Rightarrow \quad y(x_0 + h) = y_0 + hf(x_0, y_0) \quad \text{[from (1)]}$$

$$\Rightarrow \quad y(x_1) = y_0 + hf(x_0, y_0) \quad [\text{where } x_1 = x_0 + h]$$

This is an approximate solution of the equation (1) at $x = x_1$.

Denoting $y(x_1)$ by y_1, we have

$$y_1 = y_0 + hf(x_0, y_0)$$

Now, taking $x = x_1$ in (2), noting that $y(x) = y_1, y'(x) = f(x_1, y_1)$ we get

$$y(x_1 + h) = y(x_1) + h y'(x_1)$$

Setting $x_1 + h = x_2$ and $y(x_2) = y_2$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

This gives us an approximate solution of the equation (1) at $x = x_2 = x_1 + h = x_0 + 2h$.

Repeating this process n times, we get

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

which is an approximate solution at $x_n = x_0 + nh$.

This process is called **Euler's method** of finding an approximate solution of the equation (1).

Modified Euler's Method

Consider the equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y = y_0$ when $x = x_0$

The first approximation for y at $x = x_1$, by Euler's method is

$$y_1 = y_0 + hf(x_0, y_0)$$

The first modified value of y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0 + f(x_1, y_1)]$$

The second modified value of y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

We repeat this step till two consecutive values of y agree. This value is taken as y_1. Then y_2 is given by

$$y_2 = y_1 + hf(x_1, y_1)$$

Again the previous steps are repeated till y_2 becomes stationary. Then we go over to calculate y_3 as above and so on.

This is the **modified Euler's method.**

Example 1. Using Euler's method solve $\frac{dy}{dx} = 1 + xy$ with $y(0) = 2$. Find $y(0.1)$, $y(0.2)$, $y(0.3)$.

Solution : We have the Euler's formula for the differential equation $\frac{dy}{dx} = f(x, y)$

as $\quad y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \; n = 1, 2, 3, \ldots$

Here, $\quad f(x, y) = 1 + xy, \quad x_0 = 0, \; y_0 = 2, h = 0.1$

Now, $\quad y(0.1) = y_1 = y_0 + hf(x_0, y_0) = 2 + (0.1)f(0, 2) = 2.1$

Now, $\quad x_1 = x_0 + h = 0.1$, we get

$$y(0.2) = y_2 = y_1 + hf(x_1, y_1) = 2.1 + (0.1)[1 + (0.1)(2.1)] = 2.221$$

Now, $\quad x_2 = x_1 + h = 0.2$, we get

$$y(0.3) = y_3 = y_2 + hf(x_2, y_2) = 2.221 + (0.1)[1 + (0.2)(2.221)] = 2.3654$$

Example 2. Using Euler's modified method solve $\frac{dy}{dx} = x^2 + y$ where $y = 0.94$ when $x = 0$, for $x = 0.1$.

Solution : Here $\frac{dy}{dx} = f(x, y) = x^2 + y,\ x_0 = 0,\ y_0 = 0.94, h = 0.1.$

Let $x_1 = 0.1$. To find $y_1 = y(0.1)$, we have from Euler's method

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\,(x_0^2 + y_0)$$

$$\Rightarrow \quad y_1 = 0.94 + (0.1)\,[0 + 0.94] = 1.034.$$

By Euler's modified method, we have

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$\Rightarrow \quad y_1^{(1)} = 0.94 + \frac{0.1}{2}[x_0^2 + y_0 + x_1^2 + y_1]$$

$$\Rightarrow \quad y_1^{(1)} = 0.94 + \frac{0.1}{2}[0 + 0.94 + (0.1)^2 + 1.034]$$

$$\Rightarrow \quad y_1^{(1)} = 0.94 + (0.05)(1.984) = 1.0392$$

Now, $$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0 + f(x_1, y_1^{(1)})]$$

$$\Rightarrow \quad y_1^{(2)} = 0.94 + \frac{0.1}{2}[0 + 0.94 + (0.1)^2 + 1.0392]$$

$$\Rightarrow \quad y_1^{(2)} = 0.94 + (0.5)(1.9892) = 1.03946.$$

Again, $$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0 + f(x_1, y_1^{(2)})]$$

$$\Rightarrow \quad y_1^{(3)} = 0.94 + \frac{0.1}{2}[0 + 0.94 + (0.1)^2 + 1.03946]$$

$$\Rightarrow \quad y_1^{(3)} = 0.94 + (0.5)(1.98946) = 1.039473.$$

Now, $$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0 + f(x_1, y_1^{(3)})]$$

$$\Rightarrow \quad y_1^{(4)} = 0.94 + \frac{0.1}{2}[0 + 0.94 + (0.1)^2 + 1.039473]$$

$$\Rightarrow \quad y_1^{(4)} = 0.94 + (0.5)(1.989473) = 1.03947.$$

Thus $y_1^{(3)}$ and $y_1^{(4)}$ are equal correct to five decimal places. Hence $y_1 = y(0.1) = 1.03947$.

Example 3. Use Euler's modified method to compute y for $x = 0.05$ and $x = 0.1$, given that $\frac{dy}{dx} = x + y$ with the initial condition $x_0 = 0,\ y_0 = 1$.

Solution : Here $\frac{dy}{dx} = f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1,\ h = 0.5.$

Let $x_1 = 0.05.$

We have from Euler's method

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\,(x_0 + y_0)$$

$$\Rightarrow \quad y_1 = 1 + (0.05)\,[0 + 1] = 1.05.$$

By Euler's modified method, we have

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$\Rightarrow \quad y_1^{(1)} = 1 + \frac{0.05}{2}[x_0 + y_0 + x_1 + y_1]$$

$$\Rightarrow \quad y_1^{(1)} = 1 + 0.025\,[0 + 1 + 0.05 + 1.05]$$

$$\Rightarrow \quad y_1^{(1)} = 1 + (0.025)\,(2.1) = 1.0525$$

Now, $$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$\Rightarrow \quad y_1^{(2)} = 1 + \frac{0.05}{2}[0 + 1 + 0.05 + 1.0525]$$

$$\Rightarrow \quad y_1^{(2)} = 1 + (0.025)(2.1025) = 1.05256.$$

Thus $y_1^{(1)} = y_1^{(2)}$. Hence we take $y_1 = y(0.05) = 1.0526$

Now, let $x_2 = 0.1$. We have by Euler's modified method

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h\,(x_1 + y_1)$$

$$\Rightarrow \quad y_2 = 1.0526 + (0.05)\,[0.05 + 1.0526]$$

$$\Rightarrow \quad y_2 = 1.0526 + 0.05513 = 1.10773$$

Again by Euler's modified method,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1 + f(x_2, y_2)]$$

$$\Rightarrow \quad y_2^{(1)} = 1.0526 + \frac{0.05}{2}[0.05 + 1.0526 + 0.1 + 1.10773]$$

$$\Rightarrow \quad y_2^{(1)} = 1.0526 + 0.025\,(2.31033) = 1.110358 \approx 1.1104$$

Now, $$y_2^{(2)} = y_{01} + \frac{h}{2}[f(x_1, y_1 + f(x_2, y_2^{(1)})]$$

$$\Rightarrow \quad y_2^{(2)} = 1.0526 + \frac{0.05}{2}[0.05 + 1.0526 + 0.1 + 1.1104]$$

$$\Rightarrow \qquad y_2^{(2)} = 1.0526 + (0.025)(2.313) = 1.11042.$$

Here $y_2^{(1)} = y_2^{(2)}$. Thus $y_2 = y(0.1) = 1.1104$.

EXERCISES

Using Euler's modified method solve the following equations correct to 4 places of decimal.

1. $\dfrac{dy}{dx} = x + y,$ $\quad y(0) = 1$ for $x = 0.1$
2. $\dfrac{dy}{dx} = 2 + \sqrt{xy},$ $\quad y(1) = 1$ for $x = 2$, taking $h = 0.1$
3. $\dfrac{dy}{dx} = x^2 + y,$ $\quad y(0) = 1$ for $x = 0.02, 0.04$
4. $\dfrac{dy}{dx} = x^2 + y,$ $\quad y(0) = 1$ for $x = 0.2$, taking $h = 0.1$
5. $\dfrac{dy}{dx} = 1 - y,$ $\quad y(0) = 0$ for $x = 0.3$, taking $h = 0.1$.
6. $\dfrac{dy}{dx} = y - x^2,$ $\quad y(0) = 1$ in three steps with $h = 0.2$

ANSWERS

1. 1.0526   2. 5.0524   3. 1.6202 and 1.0408
4. 1.8949   5. 0.2588   6. 1.737.

1.24 Runge-Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y = y_0$ when $x = x_0$

The **Runge – Kutta method of second order** is given by

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = h.f(x_0, y_0);$ $\quad k_2 = h.f\left(x_0 + h, y_0 + k_1\right)$

The **fourth order Runge – Kutta formula** is given by

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h.f(x_0, y_0);$ $\quad k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right); \quad k_4 = h.f\left(x_0 + h, y_0 + k_3\right)$$

If we set $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ then clearly k is the weighted mean of k_1, k_2, k_3 and k_4.

Then the required approximate value is given by $y_1 = y_0 + k$.

After finding the first approximation, the approximate solution at $x_2 = x_1 + h = x_0 + 2h$ can be obtained by

$$y_1 = y_0 + k$$

where $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Here $$k_1 = h.f(x_1, y_1); \qquad k_2 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right); \qquad k_4 = h.f\left(x_1 + h, y_1 + k_3\right)$$

Solutions at $x_3 = x_2 + h$, $x_4 = x_3 + h$, ... may be obtained using the following generalized formula.

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $$k_1 = h.f(x_n, y_n); \qquad k_2 = h.f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h.f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right); \qquad k_4 = h.f\left(x_n + h, y_n + k_3\right)$$

Example 1. Solve $\frac{dy}{dx} = x + y^2$ with initial condition $y = 1$ when $x = 0$ for $x = 0.2(0.2)0.4$, using Range-Kutta method of 4th order.

Solution : Here $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$.

The first approximation of the solution is given by

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Now, $$k_1 = h.f(x_0, y_0) = (0.2)[0 + 1^2] = 0.2$$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= (0.2)f(0.1, 1.1) = (0.2)[0.1 + (1.1)^2] = 0.262$$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2)f(0.1, 1.131) = (0.2)[0.1 + (1.131)^2] = 0.2758$$

$$k_4 = h.f\left(x_0 + h, y_0 + k_3\right)$$

$$= (0.2)f(0.2, 1.2758) = (0.2)[0.2 + (1.2758)^2] = 0.3655$$

Now, $$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow \quad y_1 = 1 + \frac{1}{6}[0.2 + 2(0.262) + 2(0.2758) + 0.3655]$$

$$\Rightarrow \quad y_1 = 1 + \frac{1}{6}[0.2 + 0.524 + 0.5516 + 0.3655] = 1.2735$$

The second approximation $y_2 = y_1 + h$ is given by

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Now, $$k_1 = h.f(x_1, y_1) = (0.2)f(0.2, 1.2735)$$

$$= (0.2)[0.2 + (1.2735)^2] = 0.3644$$

$$k_2 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.2)f\left(0.2 + \frac{0.2}{2}, 1.2735 + \frac{0.3644}{2}\right)$$

$$= (0.2)[0.3 + (1.2735 + 0.1822)^2]$$

$$= (0.2)[0.3 + 2.119062] = 0.4838$$

$$k_3 = h.f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.2)f\left(0.2 + \frac{0.2}{2}, 1.2735 + \frac{0.4838}{2}\right)$$

$$= (0.2)[0.3 + (1.2735 + 0.2419)^2]$$

$$= (0.2)[0.3 + 2.2964] = 0.5193$$

$$k_4 = h.f\left(x_1 + h, y_1 + k_3\right)$$

$$= (0.2)f(0.2 + 0.2, 1.2735 + 0.5193)$$

$$= (0.2)[0.4 + (1.2735 + 0.5193)^2]$$

$$= (0.2)[0.4 + 3.21413] = 0.7228$$

Now, $$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\Rightarrow \quad y_2 = 1.2735 + \frac{1}{6}[0.3644 + 2(0.4838) + 2(0.5193) + 0.7228]$$

$\Rightarrow \quad y_2 = 1.2735 + \frac{1}{6}[0.3644 + 0.9676 + 1.0386 + 0.7228]$

$\Rightarrow \quad y_2 = 1.2735 + 0.5156 = 1.7891.$

Thus $y_2 = y(0.4) = 1.7891$ is the approximate solution of y at $x_2 = 0.4$

Example 2. Find the approximate solution at $x = 1.2$ of the equation $\frac{dy}{dx} = xy$ given $y(1) = 2$ by Range - Kutta method 4^{th} order.

Solution : Here $f(x, y) = xy$, $x_0 = 1$, $y_0 = 2$ and $h = 0.2$.

The first approximation of the solution is given by

$$y_1 = y(x_0 + h) = y(1.2) = y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Now, $k_1 = h.f(x_0, y_0) = (0.2)f(1, 2) = (0.2)(1)(2) = 0.4$

$$k_2 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)f\left(1 + \frac{0.2}{2}, 2 + \frac{0.4}{2}\right)$$

$$= (0.2)f(1.1, 2.2) = (0.2)(1.1)(2.2) = 0.484$$

$$k_3 = h.f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2)f\left(1.1, 2 + \frac{0.484}{2}\right)$$

$$= (0.2)f(1.1, 2.242) = (0.2)(1.1)(2.242) = 0.49324$$

$$k_4 = h.f\left(x_0 + h, y_0 + k_3\right)$$

$$= (0.2)f(1.2, 2.49324)$$

$$= (0.2)(1.2)(2.49324) = 0.59837$$

Now, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\Rightarrow \quad k = \frac{1}{6}[0.4 + 2(0.484) + 2(0.49324) + 0.59837]$

$\Rightarrow \quad k = \frac{1}{6}[0.4 + 0.968 + 0.98648 + 0.59837]$

$\Rightarrow \quad k = \frac{1}{6}(2.95285) = 0.49214$

$\therefore \quad y_1 = y(1.2) = y_0 + k = 2.49214$, which is the required value.

EXERCISES

Solve the following equations using Range-Kutta method

1. $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y(2) = 2$ for $x = 2.2$ taking $h = 0.1$

2. $\frac{dy}{dx} = xy^2$, $y(0) = 1$ for $x = 0.2(0.2)0.4$

3. $\frac{dy}{dx} = (x + y)^{-1}$, $y(0) = 1$ for $x = 0.5(0.5)1$

4. $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$ for $x = 0.2(0.2)0.6$

5. $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 1.5$ for $x = 1.2$, with $h = 0.1$.

ANSWERS

1. 2.4114
2. 1.0204 and 1.0869
3. 1.3571 and 1.5837
4. 0.2027, 0.4227, 0.68413
5. 2.5005.

2

LINEAR PROGRAMMING

2.1 Introduction

The linear programming (LP) is a Mathematical technique designed to optimize the usage of limited resources. Linear programming is more applicable for military, industry, agriculture, transportation, economic problems, etc.,. The computation in a linear programming involves more steps and calculations. Hence it is necessary to use the aid of computer. In this text we present the matter in such a way that the students of computer science can develop a computer program for the algorithms given.

2.2 Formulation of Linear Programming Problem

The formulation of a linear programming problem as a Mathematical Model involves the following basic steps

Step 1. Identification of the key variables on which the decision is to be taken. These variables are called **decision variables** and usually denoted by $x_1, x_2, \ldots, x_n$.

Step 2. Construction of the function Z, which involves the decision variables, to be optimizes*. The function z, which is to be decided by the decision variables $x_1, x_2, \ldots x_n$ to the best optimization is called **objective function.**

Step 3. The conditions or restriction on the choice of decision variables. The conditions are usually equalities or inequalities satisfied by the decision variables. These conditions are called **constraints.**

Step 4. The decision variables are normally non-negative reals. That is $x_i \geq 0$, for all $i = 1, 2, \ldots, n$. These are called **non-negative restrictions** (or conditions).

* Optimization is a process of maximizing or minimizing the function (objective)

The constraints are linear (i.e. no power for decision variables or product of two decision variables present) in decision variables for a linear programming problem.

The above steps can be best explained with the aid of following examples.

Example 1: Find two positive numbers whose sum is at least 10 and differences is at most 5 such that their product is maximum.

Solution:

Step 1. We have to choose two positive numbers. Let x_1 and x_2 be the required numbers. Then x_1 and x_2 become decision variables.

Step 2. Our object is to get the maximum product. The product of numbers x_1 and x_2 is $x_1 \cdot x_2$. Let $Z = x_1 \cdot x_2$. Then

$$\boxed{\text{Maximize } Z = x_1 \cdot x_2}$$ is the objective function.

Step 3. As a condition we have

(i) Sum of numbers is at least 10

i.e. $\boxed{x_1 + x_2 \geq 10}$

(ii) Difference of numbers is at most 5

i.e. $\boxed{x_1 - x_2 \leq 5}$

(iii) Since the numbers are non-negative

i.e. $\boxed{x_1 \geq 0 \text{ and } x_2 \geq 0}$

Thus, a Mathematical Model for the given problem is

Objective function

$$\text{Max } Z = x_1 \cdot x_2$$

Subject to the constraints

$$x_1 + x_2 \geq 10$$

$$x_1 - x_2 \leq 5$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Example 2. A firm manufactures headache pills in two sizes *A* and *B*. *A* constrains 2 grains of aspirin, 3 grains of bicarbonate and 2 grains of Codeine. Size *B* contains 1 grain of aspirin, 2 grains of bicarbonate and 4 grains of codeine. It is found by users that it requires at least 10 grains of aspirin, 24 grains of bicarbonate and 50 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief.

Formulate the problem as a standard L.P.P.

Solution:

Step 1. It is required to find the least number of pills of size *A* and size *B* to be taken. Thus the decision variables are number of pills of different kinds.

Let x_1 and x_2 be respectively the number of pills of size *A* and size *B* to be taken.

Step 2. As the number of pills to be taken should be minimum, the sum of pills to be minimized.

i.e. Objective function is

$$\boxed{\text{Min } Z = x_1 + x_2}$$

Step 3. To obtain the constraints we tabulate the given data's as follows:

Size	Contents (in grains)			Availability
	Aspirin	Bicarbonate	Codeine	
A	2	3	2	No restriction
B	1	2	4	No restriction
Requirement	≥ 10	≥ 24	≥ 50	

Taking x_1 number of pills of size *A* and x_2 number of pills of size *B* we get

$2x_1 + x_2$ grains of aspirin

$3x_1 + 2x_2$ grains of bicarbonate

$2x_1 + 4x_2$ grains of codeine

For immediate relief we have

Aspirin ≥ 10 ; Bicarbonate ≥ 24; Codeine ≥ 50

Hence

$$\boxed{2x_1 + x_2 \ge 10}$$

$$\boxed{3x_1 + 2x_2 \ge 24}$$

$$\boxed{2x_1 + 4x_2 \ge 50}$$

Step 4. Since one cannot consume negative numbers of pills of any kind, we have $x_1 \ge 0$ and $x_2 \ge 0$.

Summarizing all the above steps, we get the following mathematical model for the given problem

Objective function

$$\text{Minimize } Z = x_1 + x_2$$

Subject to the constraints

$$2x_1 + x_2 \ge 10$$

$$3x_1 + 2x_2 \geq 24$$

$$2x_1 + 4x_2 \geq 50$$

$$x_1 \geq 0, x_2 \geq 0.$$

Example 3. A farmer has a 100-acre land. He can sell all the vegetables *A*, *B* and *C* he can raise. The price he can obtain by selling *A*, *B* and *C* are respectively Rs. 4, Rs. 3.75 and Rs. 5 per kg. The average yield per acre of *A*, *B* and *C* are respectively 2,000, 3,000 and 1,000 kg. Fertilizer is available at Rs. 2 per kg. and the amount required per acre is 50 kg. for each *A* and *B*, and 25 kg. for *C*. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for *A* and *C*, and 6 man-days for *B*. A total of 400 man-days of labour are available at Rs. 20 per man-day.

Formulate this problem as a linear programming model to maximize the farmer's total profit.

Solution:

Let x_1, x_2 and x_3 denote respectively the number of acres the farmer has to raise the vegetables *A*, *B* and *C*. Then profit from

(i) *A* is = return from *A* – investment on *A*

= (price per kg. × number of kg. *A*) – (fertilizer + man power)

$= (\text{Rs. } 4 \times 2{,}000 \times x_1) - (\text{Rs. } 2 \times 50 \times x_1 + \text{Rs. } 20 \times 5 \times x_1)$

$= \text{Rs. } (8{,}000 - 100 - 100)\, x_1 = 7800\, x_1.$

(ii) *B* is = return from *B* – investment on *B*

= (price per kg. × number of kg. *B*) – (fertilizer + man power)

$= (\text{Rs. } 3.75 \times 3{,}000 \times x_2) - (\text{Rs. } 2 \times 50 \times x_2 + \text{Rs. } 20 \times 6 \times x_2)$

$= \text{Rs. } (11{,}250 - 100 - 120)\, x_2 = 11030\, x_2.$

(iii) *C* is = return from *C* – investment on *C*

= (price per kg. × number of kg. *C*) – (fertilizer + manpower)

$= (\text{Rs. } 5 \times 1{,}000 \times x_3) - (\text{Rs. } 2 \times 25 \times x_3 + \text{Rs. } 20 \times 5 \times x_3)$

$= \text{Rs. } (5{,}000 - 50 - 100)\, x_1 = 4850\, x_3.$

Maximum profit of the farmer is

$$\text{Max } Z = 7800\, x_1 + 11030\, x_2 + 4850\, x_3.$$

The constraints are

(i) Total man power available is 400

i.e. man power for $(A + B + C) \leq 400$

i.e. $5x_1 + 6x_2 + 5x_3 \le 400$

(ii) Total land area is 100 acre

i.e. $x_1 + x_2 + x_3 \le 100$

(iii) $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$ (non-negativity of area)

Thus, the mathematical model is

Objective function

$$\text{Max } Z = 7800x_1 + 11030x_2 + 4850x_3$$

Subject to the constraints

$$5x_1 + 6x_2 + 5x_3 \le 400$$

$$x_1 + x_2 + x_3 \le 100$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

Example 4. Formulate the following problem as a linear programming problem.

" A firm engaged in producing 2 models x_1 and x_2 performs 3-operation-painting, assembly and testing. The relevant data are as follows:

Model	Unit sale price	Hours required for each unit		
		Assembly	Painting	Testing
x_1	Rs. 50	1.0	0.2	0.0
x_2	Rs. 80	1.5	0.2	0.1

Total number of hours available are; Assembly 600, painting 100, Testing 30. Determine weekly production schedule to maximizing revenue.

Solution: Let x_1 and x_2 be respectively the number of items (units) of model x_1 and x_2 produced. Then

Profit from the production is

$$Z = \text{profit from } x_1 + \text{profit from } x_2$$

$$\boxed{Z = 50x_1 + 80x_2}$$

Requirement for x_1 and x_2 and availability is

(i) Time for assembly $\le$ 600 hours

$$(1.0)x_1 + (1.5)x_2 \le 600$$

i.e. $\boxed{x_1 + 1.5x_2 \le 600}$

(ii) Time for painting ≤ 100

i.e. $\boxed{0.2\,x_1 + 0.2\,x_2 \le 100}$

(iii) Time for testing ≤ 30hours

i.e. $\boxed{0\,x_1 + 0.1\,x_2 \le 30}$

(iv) Non-negativity in production

i.e. $\boxed{x_1 \ge 0,\, x_2 \ge 0}$

Thus, the mathematical model for the problem is

Objective function

$$\text{Max } Z = 50\,x_1 + 80\,x_2$$

Subject to the constraints

$$x_1 + 1.5\,x_2 \le 600$$

$$0.2\,x_1 + 0.2\,x_2 \le 100$$

$$0.1\,x_2 \le 30$$

$$x_1 \ge 0,\, x_2 \ge 0$$

Example 5: Formulate the following problem as a linear programming problem. Do not solve:

" Consider a small plant which makes two types of automobile parts, say *A* and *B*. It buys casting that are machined, bored and polished. The capacity of machining is 25 per hour for *A* and 40 hour for *B*. Capacity of boring is 18 per hour for *A* and 30 per hour for *B* and that of polishing is 35 per hour for *A* and 30 per hour for *B*.

Casting for *A* costs Rs. 2 each and for part *B* they cost Rs. 3 each. They sell for Rs. 5 and Rs. 6 respectively. The three machines have running costs of Rs. 20, Rs. 14 and Rs. 16 respectively per hour. Assuming that any combination of parts *A* and *B* can be sold, what product mix maximizes profit"?

Solution:

Let x_1, x_2 be the number of units of type *A* and type *B* manufactured respectively. We illustrate the data's to obtain the constraints and objective function in the following table.

Type	Cost price in Rs.	Selling price in Rs.	Working capacity of the machines per hour		
			Machining	Boring	Polishing
A	2	5	25	18	35
B	3	6	40	30	30
Running cost per hrs. in Rs.			20	14	16

Investments (in Rs.)	**For *A*** (for one unit)	**For *B*** (for one unit)
Cost price	2	3
Machining	20/25	20/40
Boring	14/18	14/30
Polishing	16/35	16/30
Net (SUM)	**4.045**	**4.50**

Profit from A by selling x_1 items = selling price – Investment

$$= 5\,x_1 - 4.045\,x_1 = 0.955\,x_1$$

Profit from B by selling x_2 items = selling price – investment

$$= 6\,x_2 - 4.5\,x_2 = 1.5\,x_2.$$

$\therefore$ Total profit $\boxed{Z = 0.955\,x_1 + 1.50\,x_2}$

To find constraints: -

Total production on machining = $\frac{x_1}{25} + \frac{x_2}{40}$ in one hours

$$\Rightarrow \boxed{\frac{x_1}{25} + \frac{x_2}{40} \le 1.}$$

Total production on Boring = $\frac{x_1}{18} + \frac{x_2}{30}$ in one hours

$$\Rightarrow \boxed{\frac{x_1}{18} + \frac{x_2}{30} \le 1}.$$

Total production on Polishing = $\frac{x_1}{35} + \frac{x_2}{30}$ in one hours

$$\Rightarrow \boxed{\frac{x_1}{35} + \frac{x_2}{30} \le 1}.$$

And one cannot produce negative number of items.

$$\Rightarrow \boxed{x_1 \ge 0, x_2 \ge 0}.$$

Thus model for the problem is:

Objective function

$$\text{Max } Z = 0.955\,x_1 + 1.50\,x_2$$

Subject to the constraints

$$\frac{x_1}{35} + \frac{x_2}{30} \le 1$$

$$\frac{x_1}{18} + \frac{x_2}{30} \le 1$$

$$\frac{x_1}{25} + \frac{x_2}{40} \le 1$$

$$x_1 \ge 0, x_2 \ge 0.$$

Example 6. An oil company produces two grades of P and Q, which are sold at Rs. 18 and Rs. 21 per gallon. The refinery can buy 2 different crudes with the following analysis and costs

Crudes	Composition			Price per gallon in rupees
	A	B	C	
1	80%	10%	10%	14
2	30%	40%	30%	10

The Rs. 21 grade must have at least 50% of A and not more than 25% of C. The Rs. 18 grade must not have more than 25% of C. In the blending process 2% of A and 1% of B and C lost because of evaporation.

Formulate the problem with a view to determine the relative amount of crude's to be mixed so that the profit is maximized.

Solution:

Let p_1 and p_2 be respectively amounts of crudes 1 and 2 mixed to produce gasoline of grade P in gallons.

Let q_1 and q_2 be respectively amounts of crudes 1 and 2 mixed to produce gasoline of grade Q in gallons.

Profit from P = (selling price of the composition) – (cost of the composition)

$= 18\ (p_1 + p_2)$ – (investment + loss in mixing)

$= 18\ (p_1 + p_2) - [(14\, p_1 + 10\, p_2) + (14 \times 80\% \times 2\%\, p_1 + 10 \times 30\% \times 2\%\, p_2) + (14 \times 10\% \times 1\% \times p_1 + 10 \times 40\% \times 1\%\, p_2) + (14 \times 10\% \times 1\%\, p_1 + 10 \times 30\cdot\% \times 1\%\, p_2)]$

$$= \left(18 - 14 - 14 \times \frac{80}{100} \times \frac{2}{100} - 4 \times \frac{10}{100} \times \frac{1}{100} - 14 \times \frac{10}{100} \times \frac{1}{100}\right) p_1$$

$$+ \left(18 - 10 - 10 \times \frac{30}{100} \times \frac{2}{100} - 10 \times \frac{40}{100} \times \frac{1}{100} - 10 \times \frac{30}{100} \times \frac{1}{100}\right) p_2$$

$= 3.758\, p_1 + 7.87\, p_2$.

Profit from Q = (selling price of the composition) – (cost of the composition)

$= 21\ (q_1 + q_2)$ – (investment + loss in mixing)

$= 21\ (q_1 + q_2) - [(14\, q_1 + 10\, q_2) + (14 \times 80\% \times 2\%\, q_1 + 10 \times 30\% \times 2\%\, q_2) + (14 \times 10\% \times 1\% \times q_1 + 10 \times 40\% \times 1\%\, q_2) + (14 \times 10\% \times 1\%\, q_1 + 10 \times 30\ \% \times 1\%\, q_2)]$

$$= \left(21 - 14 - 14 \times \frac{80}{100} \times \frac{2}{100} - 4 \times \frac{10}{100} \times \frac{1}{100} - 14 \times \frac{10}{100} \times \frac{1}{100}\right) p_1 \ +$$

$$\left(121 - 10 - 10 \times \frac{30}{100} \times \frac{2}{100} - 10 \times \frac{40}{100} \times \frac{1}{100} - 10 \times \frac{30}{100} \times \frac{1}{100}\right) p_2$$

$= 6.758\, q_1 + 10.87\, q_2$.

Therefore, total profit is z = profit from P + profit from Q

$$= 3.758\,p_1 + 7.87\,p_2 + 6.758\,q_1 + 10.87\,q_2$$

To find constraints: -

(i) Rs. 21 grade (grade Q) must have at least 50 % of A

i.e. Contents of A in $Q \geq 50$ % of the contents

i.e. $(80\%\,p_1 - 2\%\,80\%\,p_1) + (30\%p_2 - 2\%30\%p_2) \geq 50\%(p_1 + p_2)$

i.e. $\frac{80}{100}\left(1-\frac{2}{100}\right)p_1 + \frac{30}{100}\left(1-\frac{2}{100}\right)p_2 \geq \frac{50}{100}(p_1 + p_2)$

i.e. $80 \times 98\,p_1 + 30 \times 98\,p_2 \geq 5000\,(p_1 + p_2)$

i.e. $[80 \times 98 - 5000]\,p_1 + [30 \times 98 - 5000]\,p_2 \geq 0$

i.e. $2840\,p_1 - 2060\,p_2 \geq 0$

i.e. $710\,p_1 - 515\,p_2 \geq 0.$

(ii) Rs. 21 grade (grade Q) must have at least 25 % of C

i.e. Contents of C in $Q \leq 25$ % of the contents

i.e. $\frac{10}{100}\left(1-\frac{1}{100}\right)p_1 + \frac{30}{100}\left(1-\frac{1}{100}\right)p_2 \leq \frac{25}{100}(p_1 + p_2)$

i.e. $10 \times 99\,p_1 + 30 \times 99\,p_2 \leq 2500\,(p_1 + p_2)$

i.e. $[990 - 2500]\,p_1 + [2970 - 2500]\,p_2 \leq 0$

i.e. $-1510\,p_1 + 470\;p_2 \leq 0$

i.e. $151\,p_1 - 47\,p_2 \geq 0.$

(iii) Rs. 18 grade (grade P) must not have more than 20 % of C

i.e. Contents of C in $P \leq 20$ % of the contents

i.e. $\frac{10}{100}\left(1-\frac{1}{100}\right)q_1 + \frac{30}{100}\left(1-\frac{1}{100}\right)q_2 \leq \frac{20}{100}(q_1 + q_2)$

i.e. $10 \times 99\,q_1 + 30 \times 99\,q_2 \leq 2000\,(q_1 + q_2)$

i.e. $[990 - 2000]\,q_1 + [2970 - 2000]\,q_2 \leq 0$

i.e. $-1010\,p_1 + 970\;p_2 \leq 0$

i.e. $101\,p_1 - 97\,p_2 \geq 0.$

(iv) As one cannot mix negative amounts of oils, the additional restriction is

$p_1 \geq 0, p_2 \geq 0, q_1 \geq 0$ and $q_2 \geq 0.$

Thus, Mathematical model for the problem is

Objective function

$$\text{Max } Z = 3.758\, p_1 + 7.87\, p_2 + 6.758\, q_1 + 10.87\, q_2$$

Subject to the constraints

$$710\, p_1 - 515\, p_2 \geq 0$$

$$151\, p_1 - 47\, p_2 \geq 0$$

$$101\, q_1 - 97\, q_2 \geq 0$$

$$p_1 \geq 0, p_2 \geq 0, q_1 \geq 0, q_2 \geq 0.$$

EXERCISES

1. Mr. Krishnamurthy, retired Govt. officer, has received retirement benefits, viz., provident fund, gratuity, etc. He is contemplating as to how much funds he should invest in various alternatives open to him so has to maximize return on investment. The investment alternatives are- government securities, fixed deposits of a public limited company, equity shares, time deposits in banks, National savings Certificates and real estate. He has made a subjective estimate of the risk involved on a five-point scale. The data on the return on investment, the number of years. for which the funds will blocked to earn this return on investment and the subjective risk involved are as follows:

	Return	Number of years.	Risk
Government securities	6%	15	1
Company deposits	15%	3	3
Equity shares	20%	6	7
Time deposits	10%	3	1
National savings certificates	12%	6	1
Real estate	25%	10	2

He was wondering what percentage of funds he should invest in each alternative so as to maximize the return on investment. He decided that the average risk should not be more than 4, and funds should not be locked-up for more than 15 years. He would necessary invest at least 30% in real estate. Formulate an L.P model for the problem.

2. "A company produces two types of models M_1 and M_2. Each M_1 model requires 4 hours grinding and 2 hours of polishing, whereas each M_2 model requires 2 hours grinding and 5 hours of polishing. The company has 2 grinders. and 3 polishers. Each grinder works for 40 hours a week and each polisher work for 60 hours a week. Profit on M_1 model is Rs. 3.00 and on an M_2 model is Rs. 4.00. Whatever is produced in a week is sold in the market. How should the company allocate its production capacity to two types of models so that it may make the maximum profit in a week?" Formulate this problem as a linear programming problem

3. The marketing department of Everest Company has collected information on the problem of advertising for its products. This relates to the advertising media available, the number of families expected to be reached with each alternative, cost per advertisement, the maximum availability of

each medium and expected exposure of each one (Measured as the relative value of the one advertisement in each media):
The information is as given here:

Advertising media	No. of families to cover	Cost/ad. (Rs.)	Maximum availability (No. of times)	Expected exposure (Units)
TV (30 sec)	3000	8000	8	80
Radio (15 sec)	7000	3000	30	20
Sunday Edition (1/4 page)	5000	4000	4	50
Magazine (1 page)	2000	3000	2	60

Other information and requirements:

a. The advertising budget is Rs. 70,000.

b. At least 40,000 families should be covered.

c. At least 2 insertions be given in Sunday edition of Daily but not more than 4 ads should be given on the TV.

Draft this as a linear programming problem. The company's objective is to maximize the expected exposure.

4. Formulate the following problem as a linear programming problem:

"A company makes two varieties, Alpha and Beta, of pens. Each Alpha pen needs twice as much labour time as a Beta pen. If only Beta pens are manufactured the company can make 500 pens per day. The market can take only up to 150 Alpha pens and 250 Beta pens per day. If Alpha and Beta pens yield profit of Rs.8 and Rs.5 respectively per pen, determine the number of Alpha and Beta pens to be manufactured per day so as to maximize the profit".

5. A manufacturer uses three raw products a, *b*, *c* priced at Rs.30, 50, 120 per kg. Respectively. He can make three different products *A*, *B*, *C* which can be sold at Rs. 90, 100, 120 per kg. respectively. The raw products can be obtained in limited quantities, viz., 20, 15 and 10 kg. per day. Given: 2 kg. of *a* plus 1kg.of *b* plus 1kg. of *c* will yield 4kg. of *A* ; 3 kg. of *a* plus 2 kg. of *b* plus 2 kg. Of *c* will yield 7 kg. of *B*; 2 kg. Of *b* plus 1 kg. of *c* will yield 3 kg. of *C*. Formulate this as a linear programming problem.

6. A farmer has 1,000 acres of land on which he can grow corn, wheat or soybean. Each acre of corn cost Rs. 100 for preparation, requires 7 man-days of work and yields a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soybeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. The farmer has Rs. 1,00,000 for preparation and 8,000 man-days of work. Set up the linear programming equation for the problem.

7. Mr. Suresh must work at least 20 hours a week to supplement his income while attending school. He has the opportunity to work in two retail stores: in store 1, Suresh can work between 5 and 12 hours a week, and in store 2, he is allowed to work between 6 and 10 hours. Both stores pay the same hourly wage. Suresh thus wants to base his decision about how many hours to work in each store on a different criterion: work stress factor. Based on interviews with present employees, Suresh estimates that, on a scale of 1 to 10, the stress factors are 8 and 6 at sores 1 and 2 respectively. Because stress mounts by the hour, he presumes that the total stress at the end of the week is proportional to the number of hours he works in the store. Formulate the problem as a standard L.P.P.

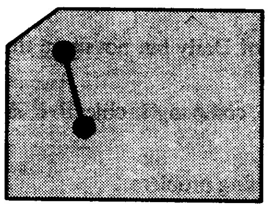
2.3 Graphical Method

In previous section we learned to construct a Mathematical Model for a given problem. We now study different methods to solve linear programming problems. The solutions are classified into various categories.

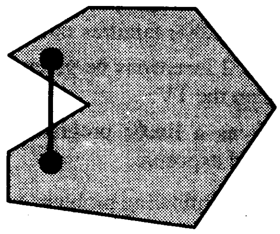
Classification of solutions:

Any non-negative decision variable that satisfies all the constraints of the model is called **feasible solution**. The region formed by the set of all feasible solution is a **feasible region**. A feasible solution that optimizes the objective function is called **optimal solution** or **optimal feasible solution**. The value of the objective function at an optimal solution is called **optimal value** of the given model.

The set of all feasible solutions of a given L.P.P. is always a **convex set*** or a convex region. The optimal solution for a L.P.P. always lies in the corner (or vertex) of the convex set.



Convex set



not a convex set

Basic steps in Graphical method:

The graphical method is applicable to solve the L.P.P. involving two-decision variables x_1 and x_2. We usually take these decision variables as x and y instead of x_1 and x_2. The graphical method includes the following steps:

Step 1. The determination of the solution space (feasible region) that defines the feasible solutions that satisfy all the constraints of the model.

Step 2. The determination of the optimal solution (corner point) from among all the points in the feasible solution space.

Types of feasible regions:

There are four types of feasible regions normally we get,

Type 1. The region is totally empty: there is no solution to the problem.

Type 2. The region contains exactly one point: the point is the optimal solution and unique.

Type 3. The region is a bounded convex set consisting of more than one point: Choose the point lies in the corner of the region. Among all the corner point choose the

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* A set S is said to be convex set if all the points lies in a line joining any two points of S belongs to S. On the other hand the line joining any two points on the region of S entirely lies in the region.

one that gives optimal value for the objective function. In case, if two optimal solutions (having the same optimal value) we get, then any point lies in the line segment joining these two optimal solutions is again a optimal solution. Hence we get infinite solutions in this case.

Type 4. The region unbounded: optimal solution may be unbounded or to be chosen by considering all corner points and comparing with the point on the region (determining the direction of the flow).

Solution of a maximization model:

The general form of the model of L.P.P. is

Objective function

$$\text{Max } Z = a\,x + b\,y$$

Subject to the constraints

$$a_i x + b_i y \le c_i \qquad i = 1, 2, \ldots, m$$

$$a_j x + b_j y \ge c_j, \qquad j = m+1, m+2, \ldots, n$$

$$x \ge 0 \text{ and } y \ge 0.$$

To solve the problem in graphical method

Step 1. Consider only the first quadrant of xy-plane (corresponds to non-negative restriction).

Step 2. For each i, $1 \le i \le n$

Draw the straight line

$$a_i x + b_i y = c_i \qquad \ldots\ldots\ldots\ldots\ldots(*)$$

by choosing two points, one in x-axis (by putting $y = 0$) and one in y-axis (putting $x = 0$) which satisfy (*) whenever $c_i \ne 0$. If $c_i = 0$, then one of the point is (0, 0) (since (*) passes through the origin) and the second required point can be chosen by taking any (positive) value for x (or y).

The line (*), divides the first quadrant into two regions, say R_1 and R_2. Choose a point $(x_1, 0)$ in the region R_1. If $(x_1, 0)$ satisfies the given inequality $a_i x + b_i y \le c_i$ (or $\ge c_i$ whichever is given), then the region R_1 is the region corresponding to the inequality constraint, shade the region R_1. Otherwise if $(x_1, 0)$ does not satisfy the inequality, then R_2 is the required region, shade the region R_2.

Step 3. Find the intersection (common) of all the shaded regions (shading should be clearly distinct for each i in step 2). The intersection of the regions is the feasible region.

Step 4. The vertices (corner points) of the feasible region obtained in step 3 are the intersection of lines corresponding to the constraints. Each vertex can be obtained by solving the corresponding equation of the lines. Find all the corner points.

Step 5. At each corner point, compute the value of the objective function.

Step 6. Identify the corner point at which the value of the objective function is **maximum**. The co-ordinate of this vertex is the optimal solution and the value of Z is the optimal value.

Example 1. Kiran is an aspiring freshman at Bangalore University. He realizes that "all works and no play make Jack a dull boy". As a result, Kiran wants to apportion his available time of about 10 hours a day between work and play. He estimate that play is twice as much as work. He also wants to study at least as much as he plays. However, Kiran realizes that if he is going to get all his homework assignment done, he cannot play more than 4 hours a day. How should Kiran allocate his time to maximize his pleasure from both work and play?

Solution: Let x and y be the time Kiran should spend to get the pleasure by work and play respectively.

As the pleasure from play he gets twice as much as that of work, total pleasure is

$$Z = x + 2y$$

Total availability of the time is 10

$$\therefore \quad x + y \leq 10$$

As he has to study at least as much as he plays

$$x \geq y \quad \Rightarrow \quad x - y \geq 0$$

The maximum time Kiran can play is 4 hours

$$\text{i.e.} \quad y \leq 4$$

Non-negativity of the decision variables

$$x \geq 0, y \geq 0$$

Thus mathematical model is

$$\text{Max } Z = x + 2y$$

Subject to the constraints

$$x + y \leq 10 \; ; \quad x - y \geq 0 \; ; \quad 0 \leq y \leq 4, x \geq 0.$$

<u>To find the region of the constrains $x + y \leq 10$</u>

$x + y = 10$:

x	0	10
y	10	0